Strategy Synthesis for Linear Arithmetic Games

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  - $\forall$ event_1, $\exists$ response_1 s.t. avoid bad state *and*
    $\forall$ event_2, $\exists$ response_2 s.t. avoid bad state *and*
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This paper:

> Algorithms for synthesizing winning strategies for satisfiability and reachability games in the theory of linear arithmetic.

Satisfiability games

Game interpretation

$$\varphi \triangleq \underbrace{\exists w. \forall x. \exists y. \forall z.}_{\textit{quantifier prefix}} \underbrace{(y < 1 \lor 2w < y) \land (z < y \lor x < z)}_{\textit{matrix}}$$

- Two players: SAT and UNSAT
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$$[\qquad\qquad\qquad\qquad]$$

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$$[w \mapsto 1; x \mapsto \frac{2}{3}; y \mapsto -1; \qquad]$$

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$$[w \mapsto 1; x \mapsto \frac{2}{3}; y \mapsto -1; z \mapsto 1]$$

  The SAT player wins if the corresponding structure is a model of the matrix.
- φ is satisfiable $\iff$ SAT has a winning strategy

$$\forall x.\forall y.\exists lub.\ \underbrace{lub \geq x \wedge lub \geq y}_{\text{upper bound}} \wedge \underbrace{[\forall ub.(ub \geq x \wedge ub \geq y) \implies ub \geq lub]}_{\text{least}}$$

$$\forall x. \forall y. \exists \mathit{lub}. \forall \mathit{ub}. \mathit{lub} \geq x \land \mathit{lub} \geq y \land [(\mathit{ub} \geq x \land \mathit{ub} \geq y) \implies \mathit{ub} \geq \mathit{lub}]$$

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Winning strategy:

$$\mathit{lub}(x, y) = \textbf{if } x \geq y \textbf{ then } x \textbf{ else } y$$

SimSat: SAT via mutual strategy improvement

[Farzan & Kincaid, IJCAI 2016]

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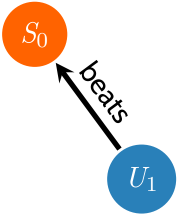

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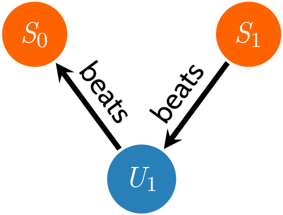

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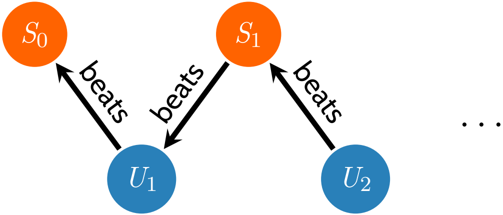

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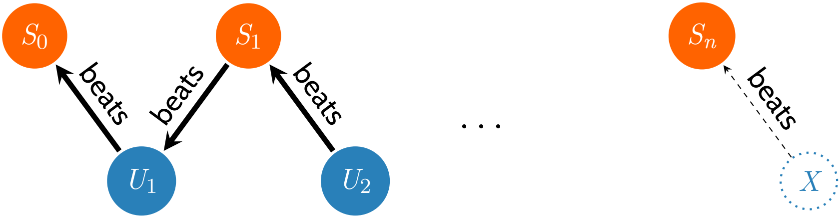

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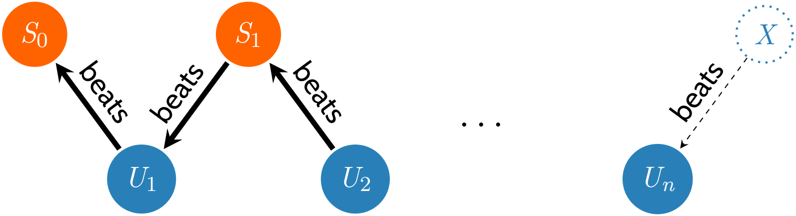

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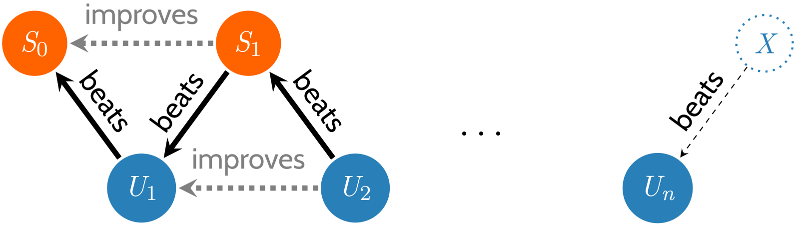

Strategy skeletons

$$\forall x. \forall y. \exists \mathit{lub}. \forall \mathit{ub}. \mathit{lub} \geq x \wedge \mathit{lub} \geq y \wedge [(\mathit{ub} \geq x \wedge \mathit{ub} \geq y) \implies \mathit{ub} \geq \mathit{lub}]$$

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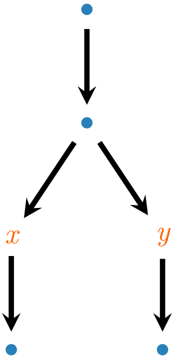

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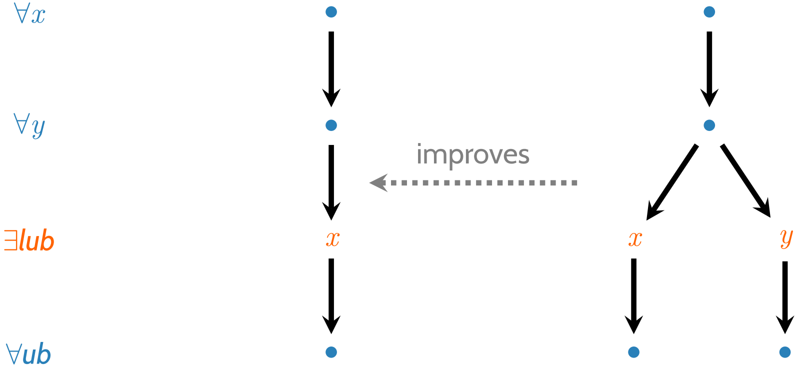
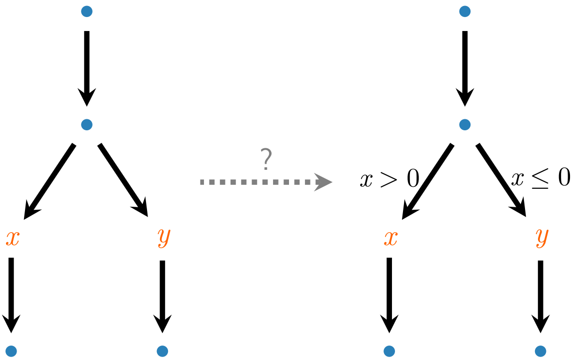


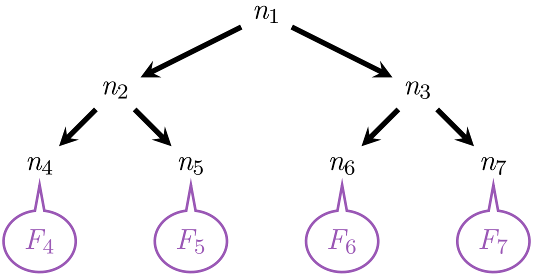
From skeletons to strategies

$$\forall x.\forall y.\exists lub.\forall ub.lub \geq x \land lub \geq y \land [(ub \geq x \land ub \geq y) \implies ub \geq lub]$$



Tree interpolation (special case)

Given tree with leaves labeled by formulas s.t. the conjunction of all labels is inconsistent:



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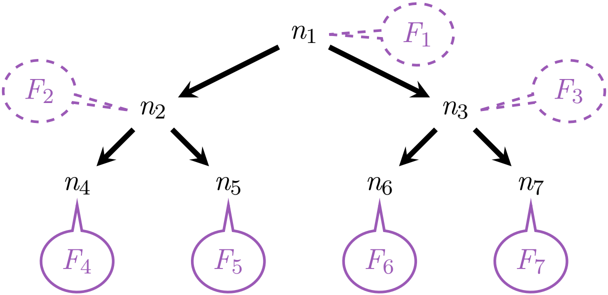

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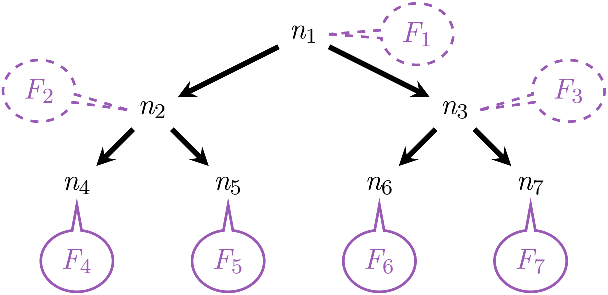

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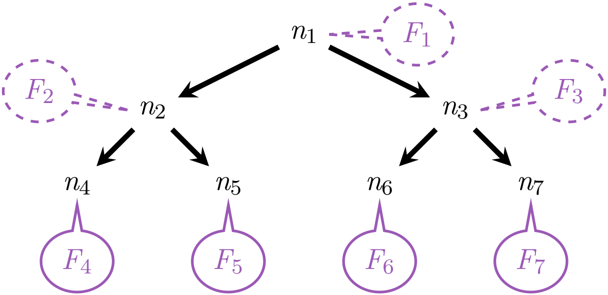

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  - conjunction of children's labels implies F_i

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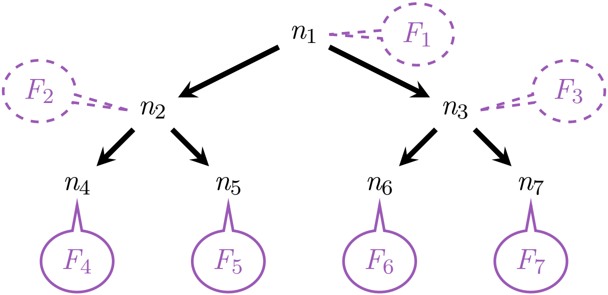

We can find labels for internal nodes s.t.:

- label of root is *false*
- For all nodes $n_i : F_i$
  - conjunction of children's labels implies F_i
  - F_i uses only symbols common to descendents & non-descendents

Strategy synthesis

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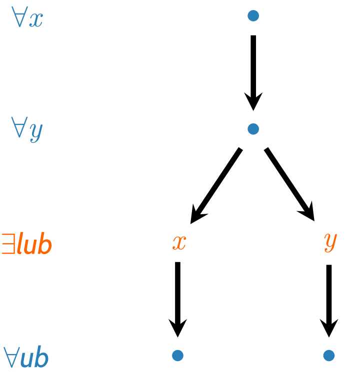

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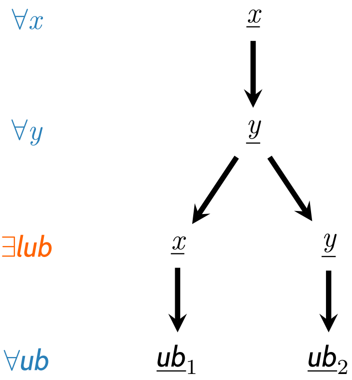

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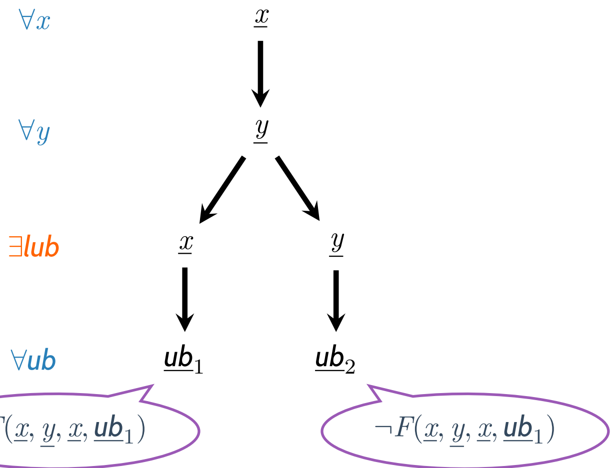

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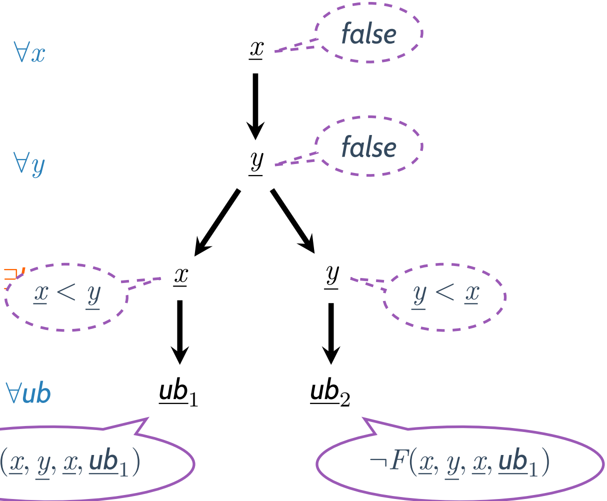

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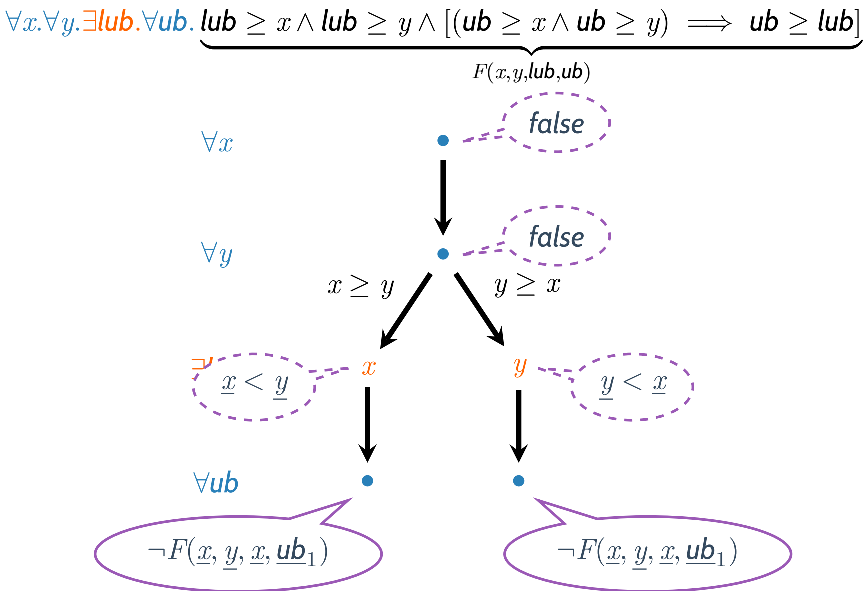


Experiments

Name	Alchemist-CSDT	CVC4-1.5.1	SimSynth
max15	Timeout	3.3s	Timeout
array_search15	Timeout	0.1s	3.0s
array_sum8_15	Timeout	0.0s	0.3s
tenfunc2	0.0s	0.1s	0.1s
polynomial4	0.0s	21.0s	0.0s
hms	Timeout	Timeout	0.0s
scaleweights	Timeout	0.1s	0.3s
lub10	Timeout	38.1s	4.0s
inverse10	Timeout	Timeout	2.4s
round10	Error	Timeout	8.8s
puzzle35	Timeout	Timeout	0.1s
puzzle35_opt	Timeout	Unknown	0.2s

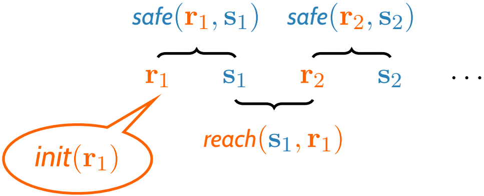
Reachability games

Definition

A *reachability game of dimension* d consists of three formulas:

- *init*$(x_1, \ldots, x_d)$: initial game state (chosen by REACH)
- *reach*$(x_1, \ldots, x_d, x'_1, \ldots, x'_d)$: moves of REACH
- *safe*$(x_1, \ldots, x_d, x'_1, \ldots, x'_d)$: moves of SAFE

REACH and SAFE alternate picking positions in $\mathbb{Q}^d$



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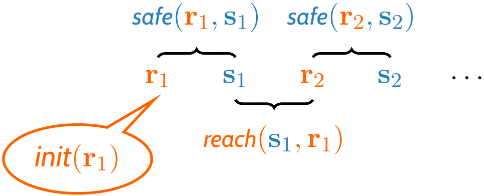
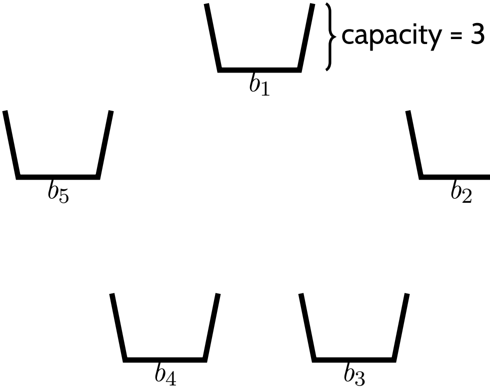


First player to make an illegal move loses.

Cinderella-Stepmother



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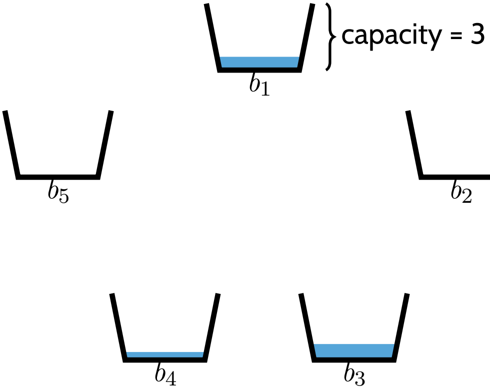

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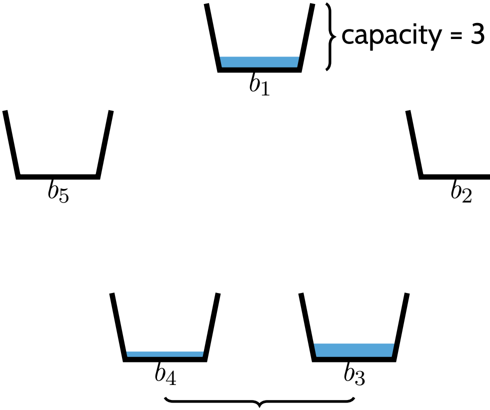

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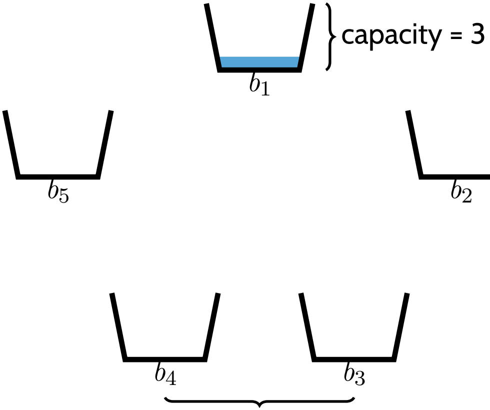


Formalizing Cinderella-Stepmother

$$\textit{init} \triangleq \left(\sum_{i=1}^{5} b_i = 1\right) \wedge \bigwedge_{i=1}^{5} b_i \geq 0$$

$$\textit{reach} \triangleq \sum_{i=1}^{5} b'_i = 1 + \left(\sum_{i=1}^{5} b_i\right) \wedge \bigwedge_{i=1}^{5} b'_i \geq b_i$$

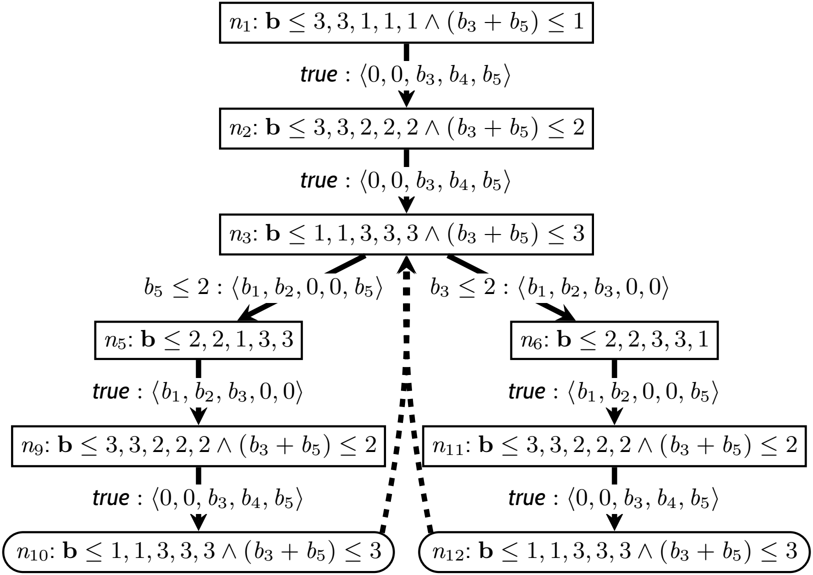
$$\textit{safe} \triangleq \neg\textit{overflow} \wedge \bigvee \textit{empty}_i$$

where

$$\textit{overflow} \triangleq \left(\bigvee_{i=1}^{5} b_i > 3\right)$$

$$\textit{empty}_i \triangleq b'_i, b'_{i+1}, b'_{i+2}, b'_{i+3}, b'_{i+4} = 0, 0, b_{i+2}, b_{i+3}, b_{i+4}$$

Safety trees



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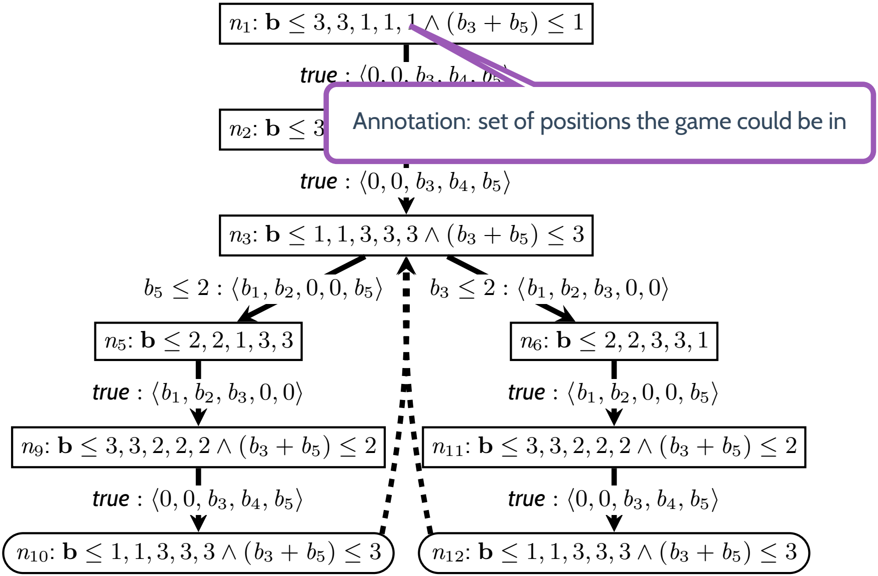

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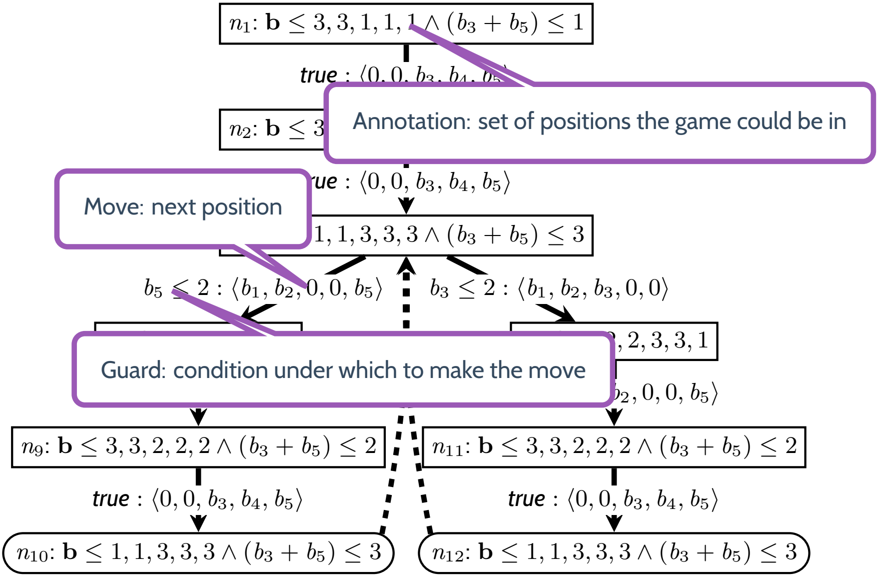
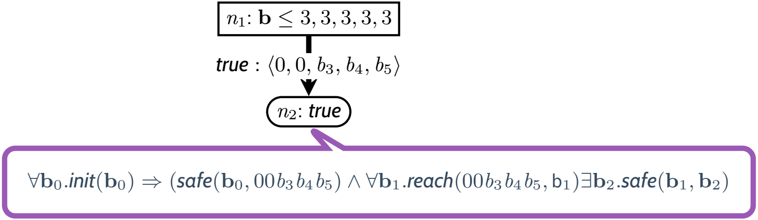
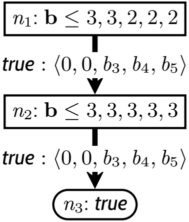


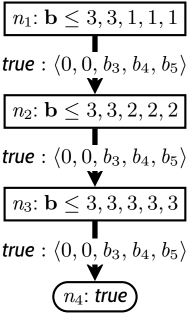
n_1: *true*

$$\forall \mathbf{b}_0.\mathit{init}(\mathbf{b}_0) \Rightarrow \exists \mathbf{b}_1.\mathit{safe}(\mathbf{b}_0, \mathbf{b}_1)$$


n_1: $\mathbf{b} \leq 3, 3, 3, 3, 3$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_2: *true*
$\forall \mathbf{b}_0 . init(\mathbf{b}_0) \Rightarrow (safe(\mathbf{b}_0, 00 b_3 b_4 b_5) \wedge \forall \mathbf{b}_1 . reach(00 b_3 b_4 b_5, \mathtt{b}_1) \exists \mathbf{b}_2 . safe(\mathbf{b}_1, \mathbf{b}_2)$



n_1: $\mathbf{b} \leq 3, 3, 2, 2, 2$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_2: $\mathbf{b} \leq 3, 3, 3, 3, 3$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_3: *true*



n_1: $\mathbf{b} \leq 3,3,1,1,1$
true : $\langle 0,0,b_3,b_4,b_5\rangle$
n_2: $\mathbf{b} \leq 3,3,2,2,2$
true : $\langle 0,0,b_3,b_4,b_5\rangle$
n_3: $\mathbf{b} \leq 3,3,3,3,3$
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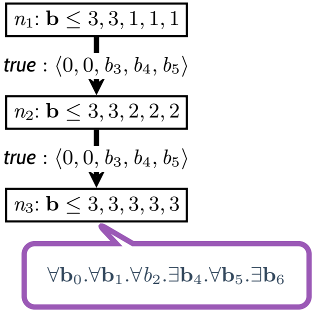
n_4: *true*



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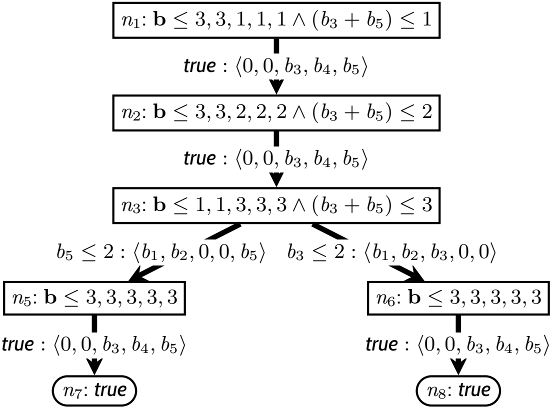
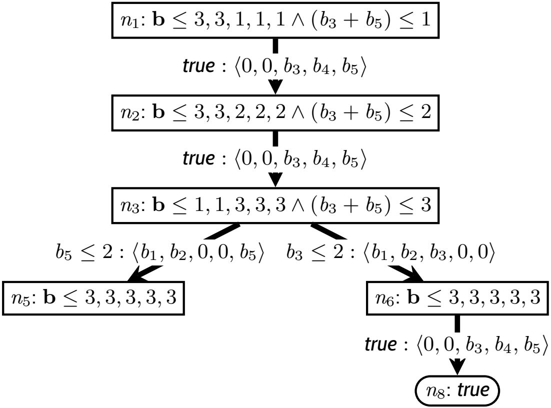
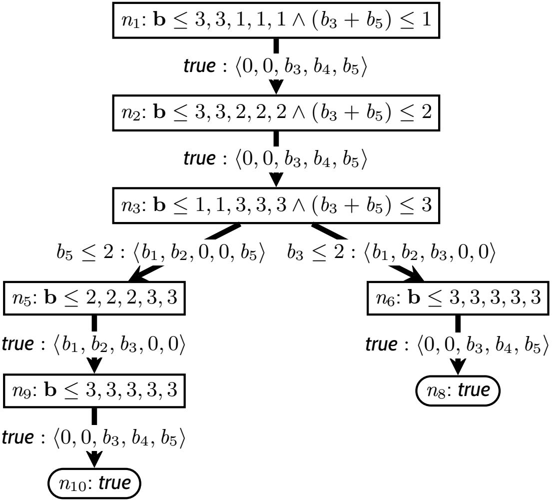
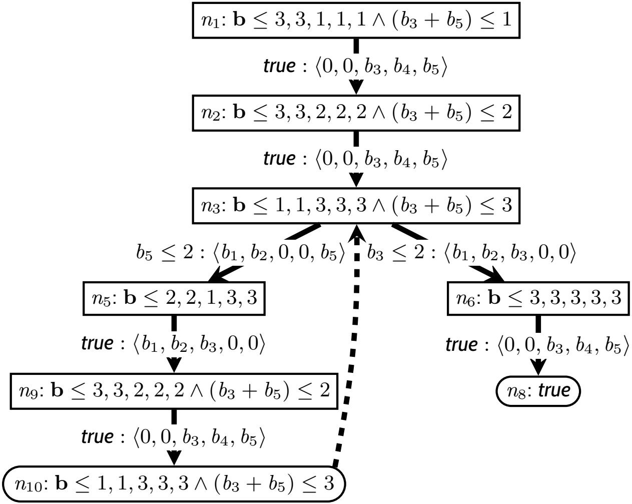
n_3: $\mathbf{b} \leq 3, 3, 3, 3, 3$
$\forall \mathbf{b}_0 . \forall \mathbf{b}_1 . \forall b_2 . \exists \mathbf{b}_4 . \forall \mathbf{b}_5 . \exists \mathbf{b}_6$


n_1: $\mathbf{b} \leq 3, 3, 1, 1, 1 \wedge (b_3 + b_5) \leq 1$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_2: $\mathbf{b} \leq 3, 3, 2, 2, 2 \wedge (b_3 + b_5) \leq 2$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_3: $\mathbf{b} \leq 1, 1, 3, 3, 3 \wedge (b_3 + b_5) \leq 3$
$b_5 \leq 2 : \langle b_1, b_2, 0, 0, b_5 \rangle$
$b_3 \leq 2 : \langle b_1, b_2, b_3, 0, 0 \rangle$
n_5: $\mathbf{b} \leq 3, 3, 3, 3, 3$
n_6: $\mathbf{b} \leq 3, 3, 3, 3, 3$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_7: *true*
n_8: *true*

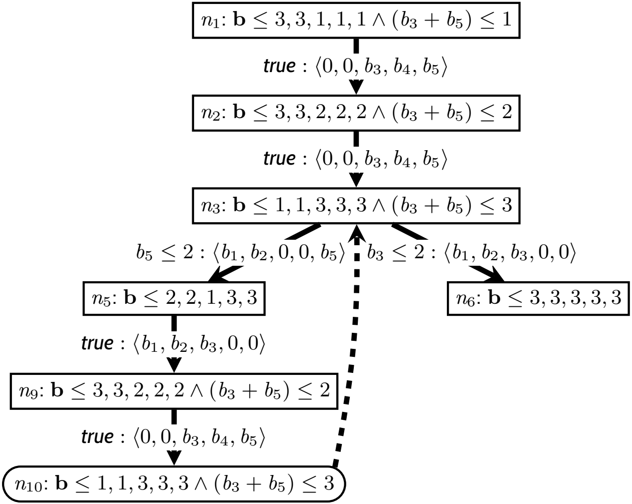


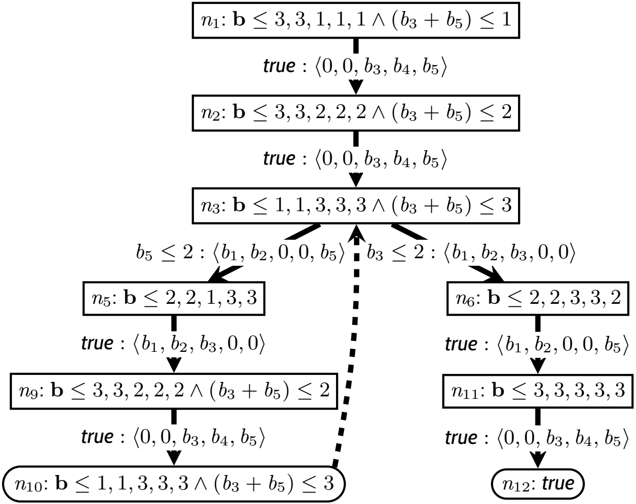
n_1: $\mathbf{b} \leq 3, 3, 1, 1, 1 \wedge (b_3 + b_5) \leq 1$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_2: $\mathbf{b} \leq 3, 3, 2, 2, 2 \wedge (b_3 + b_5) \leq 2$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_3: $\mathbf{b} \leq 1, 1, 3, 3, 3 \wedge (b_3 + b_5) \leq 3$
$b_5 \leq 2 : \langle b_1, b_2, 0, 0, b_5 \rangle$
$b_3 \leq 2 : \langle b_1, b_2, b_3, 0, 0 \rangle$
n_5: $\mathbf{b} \leq 3, 3, 3, 3, 3$
n_6: $\mathbf{b} \leq 3, 3, 3, 3, 3$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_8: *true*


n_1: $\mathbf{b} \leq 3, 3, 1, 1, 1 \wedge (b_3 + b_5) \leq 1$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_2: $\mathbf{b} \leq 3, 3, 2, 2, 2 \wedge (b_3 + b_5) \leq 2$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_3: $\mathbf{b} \leq 1, 1, 3, 3, 3 \wedge (b_3 + b_5) \leq 3$
$b_5 \leq 2 : \langle b_1, b_2, 0, 0, b_5 \rangle$
$b_3 \leq 2 : \langle b_1, b_2, b_3, 0, 0 \rangle$
n_5: $\mathbf{b} \leq 2, 2, 2, 3, 3$
n_6: $\mathbf{b} \leq 3, 3, 3, 3, 3$
true : $\langle b_1, b_2, b_3, 0, 0 \rangle$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_9: $\mathbf{b} \leq 3, 3, 3, 3, 3$
n_8: *true*
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_{10}: *true*

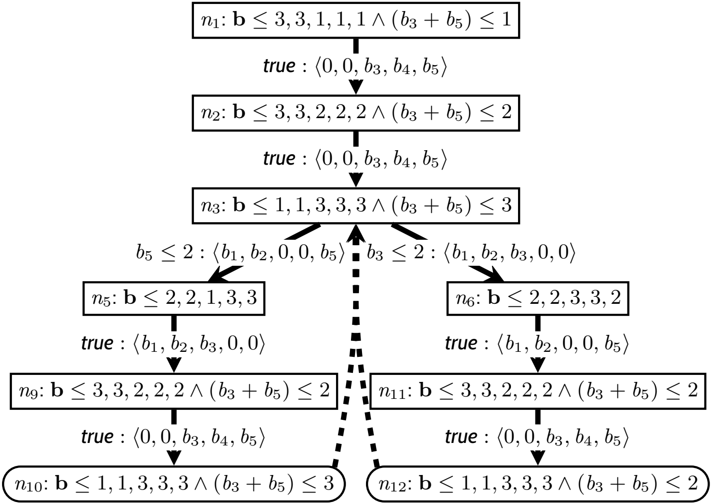


n1: b ≤ 3, 3, 1, 1, 1 ∧ (b3 + b5) ≤ 1
true : ⟨0, 0, b3, b4, b5⟩
n2: b ≤ 3, 3, 2, 2, 2 ∧ (b3 + b5) ≤ 2
true : ⟨0, 0, b3, b4, b5⟩
n3: b ≤ 1, 1, 3, 3, 3 ∧ (b3 + b5) ≤ 3
b5 ≤ 2 : ⟨b1, b2, 0, 0, b5⟩
b3 ≤ 2 : ⟨b1, b2, b3, 0, 0⟩
n5: b ≤ 2, 2, 1, 3, 3
n6: b ≤ 3, 3, 3, 3, 3
true : ⟨b1, b2, b3, 0, 0⟩
true : ⟨0, 0, b3, b4, b5⟩
n9: b ≤ 3, 3, 2, 2, 2 ∧ (b3 + b5) ≤ 2
n8: true
true : ⟨0, 0, b3, b4, b5⟩
n10: b ≤ 1, 1, 3, 3, 3 ∧ (b3 + b5) ≤ 3


n_1: $\mathbf{b} \leq 3, 3, 1, 1, 1 \wedge (b_3 + b_5) \leq 1$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_2: $\mathbf{b} \leq 3, 3, 2, 2, 2 \wedge (b_3 + b_5) \leq 2$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_3: $\mathbf{b} \leq 1, 1, 3, 3, 3 \wedge (b_3 + b_5) \leq 3$
$b_5 \leq 2 : \langle b_1, b_2, 0, 0, b_5 \rangle$
$b_3 \leq 2 : \langle b_1, b_2, b_3, 0, 0 \rangle$
n_5: $\mathbf{b} \leq 2, 2, 1, 3, 3$
n_6: $\mathbf{b} \leq 3, 3, 3, 3, 3$
true : $\langle b_1, b_2, b_3, 0, 0 \rangle$
n_9: $\mathbf{b} \leq 3, 3, 2, 2, 2 \wedge (b_3 + b_5) \leq 2$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_{10}: $\mathbf{b} \leq 1, 1, 3, 3, 3 \wedge (b_3 + b_5) \leq 3$



n_1: $\mathbf{b} \leq 3, 3, 1, 1, 1 \land (b_3 + b_5) \leq 1$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_2: $\mathbf{b} \leq 3, 3, 2, 2, 2 \land (b_3 + b_5) \leq 2$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_3: $\mathbf{b} \leq 1, 1, 3, 3, 3 \land (b_3 + b_5) \leq 3$
$b_5 \leq 2 : \langle b_1, b_2, 0, 0, b_5 \rangle$
$b_3 \leq 2 : \langle b_1, b_2, b_3, 0, 0 \rangle$
n_5: $\mathbf{b} \leq 2, 2, 1, 3, 3$
n_6: $\mathbf{b} \leq 2, 2, 3, 3, 2$
true : $\langle b_1, b_2, b_3, 0, 0 \rangle$
true : $\langle b_1, b_2, 0, 0, b_5 \rangle$
n_9: $\mathbf{b} \leq 3, 3, 2, 2, 2 \land (b_3 + b_5) \leq 2$
n_{11}: $\mathbf{b} \leq 3, 3, 3, 3, 3$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_{10}: $\mathbf{b} \leq 1, 1, 3, 3, 3 \land (b_3 + b_5) \leq 3$
n_{12}: *true*



n_1: $\mathbf{b} \leq 3, 3, 1, 1, 1 \wedge (b_3 + b_5) \leq 1$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_2: $\mathbf{b} \leq 3, 3, 2, 2, 2 \wedge (b_3 + b_5) \leq 2$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_3: $\mathbf{b} \leq 1, 1, 3, 3, 3 \wedge (b_3 + b_5) \leq 3$
$b_5 \leq 2 : \langle b_1, b_2, 0, 0, b_5 \rangle$
$b_3 \leq 2 : \langle b_1, b_2, b_3, 0, 0 \rangle$
n_5: $\mathbf{b} \leq 2, 2, 1, 3, 3$
n_6: $\mathbf{b} \leq 2, 2, 3, 3, 2$
true : $\langle b_1, b_2, b_3, 0, 0 \rangle$
true : $\langle b_1, b_2, 0, 0, b_5 \rangle$
n_9: $\mathbf{b} \leq 3, 3, 2, 2, 2 \wedge (b_3 + b_5) \leq 2$
n_{11}: $\mathbf{b} \leq 3, 3, 2, 2, 2 \wedge (b_3 + b_5) \leq 2$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
true : $\langle 0, 0, b_3, b_4, b_5 \rangle$
n_{10}: $\mathbf{b} \leq 1, 1, 3, 3, 3 \wedge (b_3 + b_5) \leq 3$
n_{12}: $\mathbf{b} \leq 1, 1, 3, 3, 3 \wedge (b_3 + b_5) \leq 2$

Cinderella-Stepmother

Capacity	Winner	Time
c=3	Cinderella	2.2s
c=2.5	Cinderella	53.8s
c=2	Cinderella	68.9s
c=1.8	–	Timeout
c=1.7	Stepmother	2.5s
c=1.6	Stepmother	1.5s
c=1.5	Stepmother	1.4s
c=1.4	Stepmother	0.2s

Cinderella-Stepmother

Capacity	Winner	Time
c=3	Cinderella	2.2s
c=2.5	Cinderella	53.8s
c=2	Cinderella	68.9s
c=1.8	–	Timeout
c=1.7	Stepmother	2.5s
c=1.6	Stepmother	1.5s
c=1.5	Stepmother	1.4s
c=1.4	Stepmother	0.2s

"the problem becomes more challenging for $1.5 \leq c < 3$ …in such cases fully automated strategy synthesis seems unrealistic, and computer-assisted proofs driven by user-provided hints or templates are more plausible."

– Beyene, Chaudhuri, Popeea & Rybalchenko; POPL'14

Summary

- Complete procedure for satisfiability games
  - Extends LRA decision procedure to strategy synthesis
- Semi-algorithm for reachability games
  - Synthesize strategies for bounded games, then generalize
  - Complete for finite REACH strategies