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Quick Relaxation in Collective Motion

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***Abstract*— We establish sufficient conditions for the quick relaxation to kinetic equilibrium in the classic Vicsek-Cucker-Smale model of bird flocking. The convergence time is polynomial in the number of birds as long as the number of flocks remains bounded. This new result relies on two key ingredients: exploiting the convex geometry of embedded averaging systems; and deriving new bounds on the s-energy of disconnected agreement systems. We also apply our techniques to bound the relaxation time of certain pattern-formation robotic systems investigated by Sugihara and Suzuki.**

I. INTRODUCTION

Introduced by Reynolds [11] in 1987, three heuristic rules have been used widely to produce spectacular bird flocking animations. The three flocking rules are (1) *separation*: avoid collision (2) *cohesion*: stay grouped together, and (3) *alignment*: align headings. Several models are constructed based on these rules to understand flocking dynamics.

We study a variant of the classic Vicsek-Cucker-Smale model [6], [14], a group of n birds are flying in the air while interacting via a time-varying network [1], [5], [7], [8]. The vertices of the network correspond to the n birds and any two distinct birds are joined by an edge if their distance is at most some fixed $r \leq 1$. The flocking network G_t is thus symmetric and loopless. Its connected components are the *flocks*. Each bird i has a position $x_i(t)$ and a velocity $v_i(t)$, both of them vectors in $\mathbb{R}^3$. Given the state of the system at time $t = 0$, we have the recurrence: for any $t \geq 0$,

$$\begin{cases} x_i(t+1) = x_i(t) + v_i(t+1); \\ v_i(t+1) = v_i(t) + a_i \sum_{j \in N_i(t)} \big(v_j(t) - v_i(t)\big), \end{cases} \tag{1}$$

where $N_i(t)$ is the set of vertices adjacent to i at time t. At each step, a bird adjusts its velocity by taking a weighted average with its neighbors. The weights a_i indicate the amount of influence birds exercise on their neighbors. To avoid negative weights, we require that $0 < a_i \leq 1/(|N_i(t)| + 1)$. We write $\rho := \min_i a_i \in (0, 1/2]$.

Intuitively, by repeating the recurrence, each bird should eventually converge to a fixed speed and direction. This is supported by computer simulations and several convergence results [7], [9], [10]. However, as was shown in [5], the model above might be periodic and never stabilize. To remedy this, we stipulate that, for two birds to be newly joined by an edge, their velocities must differ by at least a minimum amount: Formally, we require that, at any time t, $(i, j) \in G_t \setminus G_{t-1}$ if $\|x_i(t) - x_j(t)\| \leq r$ and $\|v_i(t) - v_j(t)\| > \varepsilon_o$, for small fixed positive ε_o. By space and time scale invariance, we may assume[1] that $\|x_i(0)\| \leq 1$ and $\|v_i(0)\| \leq \sqrt{\rho/n}$ for all birds i. We state our main result:

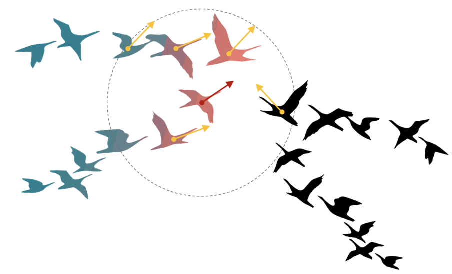

Fig. 1. A bird is influenced by its neighbors within distance r .

> THEOREM 1.1. A group of n birds forming a maximum of $m \leq n$ flocks relax to within ε of a fixed velocity vector in time $O(n^2/\rho)\log(1/\varepsilon) + t_o$, where $t_o = O\big(n^2/\rho\big)^{m+2}\log(n/\rho)$.

The main novelty of this result is that the convergence time is polynomial in the number of birds, as long as the number of flocks is bounded by a constant. The proof of the theorem relies on two key ingredients: new bounds on the s-*energy*; and the specific convex geometry of flocking. In §II, we establish new (upper and lower) bounds on the s-energy of *reversible agreement systems*. While the connected case has been well studied [3], [4], the disconnected case was wide open. We prove a nearly tight upper bound on the s-energy of such systems, which is a result of independent interest. In §III, we explore the convex geometry of flocking to bound the angle of attack between two newly joined birds as a function of time. Together with our new bounds on the s-energy, this geometric insight plays a crucial part in the proof of Theorem 1.1.

In §IV, we investigate a distributed motion coordination algorithm introduced by Sugihara and Suzuki [2], [13]. The idea is to use a swarm of robots to produce a preset pattern, in this case a polygon. We prove a polynomial bound on the relaxation time of this process. We enhance the model by allowing faulty communication and proving that the end result is robust under stochastic errors. We also generalize the geometry to 3D and arbitrary communication graphs.

II. REVERSIBLE AGREEMENT SYSTEMS

Let P_t be the stochastic matrix of a time-reversible random walk in an undirected n-vertex graph G_t. This means that

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[1]These bounds are arbitrary and the choice of $\sqrt{\rho/n}$ is made only to simplify some calculations.

$P_t = Q^{-1}M_t$, where (i) $Q = \mathrm{diag}(q)$, for $q = M_t\mathbf{1} \le \mathbf{1}/\rho$ and constant $\rho \in (0,1/2]$; and (ii) M_t is a symmetric matrix with nonzero entries at least 1 and a positive diagonal. Given $x \in \mathbb{R}^n$, the infinite sequence of vectors $(P_t \cdots P_0 x)_{t\ge 0}$ forms an orbit of a *reversible averaging system* (*RAS*). When all the matrices are the same, $P_t = P$, the map $x \mapsto Px$ is the dual map of the reversible Markov chain ($y \mapsto yP$) and its convergence time is given by the mixing time of the chain. The novelty of the model lies in the presence of time-varying matrices.

A. The s-Energy

Consider an infinite sequence of graphs $(G_t)_{t\ge0}$ with P_t the stochastic matrix of its corresponding Markov chain. Note that q is proportional to the stationary distribution of the Markov chain induced by P_t. By reversibility, we have $q_i(P_t)_{ij} = q_j(P_t)_{ji}$. Write $\langle x,y\rangle_q := \sum_i q_i x_i y_i$ and $\|x\|_q^2 := \langle x,x\rangle_q$. We call $\|x-\hat{x}\|_q^2$ the *variance* of the system, where $\hat{x} = \|q\|_1^{-1}\langle x,\mathbf{1}\rangle_q \mathbf{1}$ and x is shorthand for $x(0)$.

Given $x(0) \in \mathbb{R}^n$, we write $x(t+1) = P_t x(t)$ and we interpret $x(t)$ as an embedding of the graph G_t in $\mathbb{R}$; i.e., nodes in G_t correspond to points in the real line and edges are intervals joining two end nodes. The union of the embedded edges of G_t forms disjoint intervals, called *blocks*. Let $l_1,\dots,l_k$ be the lengths of these blocks and put $E_{s,t} = \sum_{i=1}^k l_i^s$, with $s \in (0,1]$. Following [4] , we define the *s-energy* $E_s = \sum_{t\ge0} E_{s,t}$. We denote by $\mathcal{E}_{m,s}$ the supremum of E_s over all systems of unit variance whose underlying graphs G_t have at most m connected components.

B. An Upper Bound

We prove the following bound on the s-energy of reversible agreement systems and then we show in §II-C why it is close to optimal.

Theorem 2.1. $\mathcal{E}_{m,s} \le (cn^2/\rho s)^m$, for any $s \in (0,1]$ and constant $c > 0$.

Proof. The convergence rate of the attracting dynamics is captured by a variant of the Dirichlet form: $D_t = \sum_i \max_{j:(i,j)\in G_t}\big(x_i(t) - x_j(t)\big)^2$. We omit the index t below for clarity.

Lemma 2.2. $\|Px\|_q^2 \le \|x\|_q^2 - D/2$, for any $x \in \mathbb{R}^n$.

Proof. Write $\delta_{ij} = x_i - x_j$ and $\mu_i = \sum_j p_{ij}\,\delta_{ij}$. Fix i and pick any k such that $m_{ik} > 0$. By Cauchy-Schwarz and $m_{ii} \ge 1$ (because nonzero entries of M are at least 1), we have

$$\begin{aligned}\delta_{ik}^2 &= \big((\mu_i - \delta_{ii}) + (\delta_{ik} - \mu_i)\big)^2\\ &\le 2(\delta_{ii} - \mu_i)^2 + 2(\delta_{ik} - \mu_i)^2\\ &\le 2\sum_j m_{ij}\,(\delta_{ij} - \mu_i)^2;\end{aligned}$$

hence,

$$\begin{aligned}\|x\|_q^2 - \|Px\|_q^2 &= \sum_i q_i x_i^2 - \sum_i q_i\Big(x_i + \sum_j p_{ij}\delta_{ji}\Big)^2\\ &= -\sum_i q_i\Big(2x_i\sum_j p_{ij}\delta_{ji} + \mu_i^2\Big)\\ &= \sum_i q_i\Big(\sum_j p_{ij}\,\delta_{ij}^2 - \mu_i^2\Big)\\ &= \sum_{i,j} m_{ij}\,(\delta_{ij} - \mu_i)^2 \ge \frac{1}{2}\sum_i \max_{j:m_{ij}>0}\delta_{ij}^2,\end{aligned}$$

with the last equality expressing the identity for the variance: $\mathbb{E}X^2 - (\mathbb{E}\,X)^2 = \mathbb{E}[X - \mathbb{E}\,X]^2$. □

Write $G_{\le t}$ as the union of all the edges in $G_0,\dots,G_t$, and let t_c be the maximum value of t such that $G_{\le t}$ has fewer connected components than $G_{\le t-1}$; if no such t , set $t_c = 1$.

Lemma 2.3. If $G_{\le t_c}$ is connected, $\sum_{t\le t_c} D_t \ge \rho n^{-2}\|x - \hat{x}\|_q^2$.

Proof. Let G_{t_0} denote the graph over n vertices with no edges. We define $t_1,\dots,t_c$ as the sequence of times t at which the addition of G_t reduces the number of connected components in $G_{\le t-1}$. At any time t_k ($k > 0$), the drop d_k in the number of components can be achieved by d_k edges from G_{t_k}. Let F_k denote such a set of edges: we can always order F_k so that every edge in the sequence contains at least one vertex not encountered yet. This shows that the sum of the squared lengths of the edges in F_k does not exceed D_{t_k}. We note that $F := F_1 \cup \cdots \cup F_c$ forms a collection of $n-1$ edges from (the connected graph) $G_{\le t_c}$ and F spans all n vertices.

Consider the intervals formed by the edges in F_k at time t_k, for all $k \in [c]$. The union of these intervals covers the smallest interval $[a,b]$ enclosing all the vertices at time 0 (and hence at all times). To see why, pick any z such that $a < z < b$ and denote by L and R the vertices on both sides of z at time 0. Neither set is empty and, by convexity, both of them remain on their respective side of z until an edge of some G_t joins L to R. When that happens (which it must since $G_{\le t_c}$ is connected), the joining edge(s) reduce(s) the number of components of $G_{\le t-1}$ by at least one, so F must grab at least one of them, which proves our claim. Let $l_1,\dots,l_{n-1}$ denote the lengths of the edges of F (at the time of their insertion). By Cauchy-Schwarz, $\sum_{t\le t_c} D_t \ge \sum_{i=1}^{n-1} l_i^2 \ge (b-a)^2/(n-1)$. The lemma follows from the inequalities $\|x - \hat{x}\|_q^2 \le \|q\|_1(b-a)^2 \le (n/\rho)(b-a)^2$. □

Assume that $G_{\le t_c}$ is connected. By Lemma 2.3 and the telescoping use of Lemma 2.2,

$$\|x\|_q^2 - \|x(t_c+1)\|_q^2 \ge \frac{1}{2}\sum_{t=0}^{t_c} D_t \ge \frac{\rho}{2n^2}\,\|x - \hat{x}\|_q^2\,. \qquad (2)$$

Let $U(n,m)$ be the maximum s-energy of an *RAS* with at most n vertices and m connected components at any time, subject to the initial condition $\|x - \hat{x}\|_q^2 \le 1$. By shifting the system if need be, we can always assume that $\hat{x} = \mathbf{0}$. By (2), $\|x(t)\|_q^2$ shrinks by at least a factor of $\alpha := 1 - \rho/2n^2$ by time $t_c + 1$. A simple scaling argument shows that the s-energy expanded after t_c is at most $\alpha^{s/2}U(n,m)$. While $t < t_c$ (or if $G_{\le t_c}$ is not connected), the system can be decoupled into two *RAS*, each one with fewer than m components.[2] Since

[2]Note that each subsystem satisfies the required inequalities about the Q and M entries; also, shifting each subsystem so that $\hat{x} = 0$ cannot increase $\|x\|_q^2$, so its value remains at most 1.

$\|x\|_q^2 \le 1$, the diameter at any time is at most 2; therefore $U(n,m) \le \alpha^{s/2} U(n,m) + 2U(n,m-1) + m2^s$. It follows that

$$U(n,m) \le \frac{2}{1-\alpha^{s/2}} \big(U(n,m-1) + m\big); \qquad (3)$$

hence $U(n,m) = O(n^2/\rho s)^m$, and Theorem 2.1 follows. □

C. A Lower Bound

We begin with the case $m = 1$. The path graph G over n vertices has an edge $(i, i+1)$ for all $i < n$. The Laplacian L is $\mathrm{diag}(u) - A$, where A is the adjacency matrix of G and u is the degree vector $(1, 2, \ldots, 2, 1)$. We consider the *RAS* formed by the matrix $P_t = P = I - \rho L$. By well-known spectral results on graphs [12], P_t has a full set of n orthogonal eigenvectors v_k, where $v_k(i) = \cos \frac{(i-1/2)k\pi}{n}$ for $i \in [n]$, with its associated eigenvectors $\lambda_k = 1 - 2\rho\big(1 - \cos \frac{k\pi}{n}\big)$, for $0 \le k < n$. We require $\rho < 1/4$ to ensure that P is positive semidefinite. We initialize the system with $x = (1, 0, \ldots, 0)$ and observe that the agents always keep their initial rank order, so the diameter Δ_t at time t is equal to $(1, 0, \ldots, 0, -1)P^{t-1}x$. We verify that $\|v_1\|^2 = n/2$. By the spectral identity $P^j = \sum_{k<n} \lambda_k^j v_k v_k^T / \|v_k\|^2$, we find that, for $t > 1$ and $n > 1$,

$$\begin{aligned}\Delta_t &= \sum_{k=0}^{n-1} \lambda_k^{t-1} v_k(1) \frac{v_k(1) - v_k(n)}{\|v_k\|^2} \\ &= \sum_{\text{odd } k} \frac{2\lambda_k^{t-1}}{\|v_k\|^2} \Big(\cos \frac{k\pi}{2n}\Big)^2 \\ &\ge \frac{2\lambda_1^{t-1}}{\|v_1\|^2} \Big(\cos \frac{\pi}{2n}\Big)^2 \ge \frac{2}{n} \lambda_1^{t-1}.\end{aligned}$$

The s-energy is equal to $\sum_t \Delta_t^s \ge (2/n)^s/(1-\lambda_1^s) \ge bn^{2-s}/\rho s$, for constant $b > 0$.

For the general case, we denote by $F(n,m)$ the s-energy of the system with initial diameter equal to 1. We showed that $F(n,1) \ge bn^{2-s}/\rho s$. We now describe the steps of the dynamics for $m > 1$. To simplify the notation, we assume that $\nu := n/m$ is an integer.[3] For $i \in [m]$, let C_i be the path linking vertices $[(i-1)\nu + 1, i\nu]$.

1) At time $t = 1$, the vertices of C_1 are placed at position 0 while all the others are stationed at 1. The paths C_1 and C_2 are linked together into a single path so the system has $m-1$ components. Vertices ν and $\nu+1$ move to positions ρ and $1 - \rho$ respectively while the others do not move at all. The s-energy expended during that step is equal to 1.
2) The system now consists of the m paths C_i. We apply the case $m = 1$ to C_1 and C_2 in parallel, which expends s-energy equal to $2\rho^s F(\nu, 1)$. All other vertices stay in place. The transformation keeps the mass center invariant, so the vertices of C_1 and C_2 end up at positions ρ/ν and $1 - \rho/\nu$, respectively.[4]
3) We move the vertices in C_i for $i \ge 2$ by applying the same construction recursively for fewer than m components. The vertices of C_1 stay in place. The s-energy used in the process is equal to $\big(\rho/\nu\big)^s F(n-\nu, m-1)$ and the vertices of $C_2, \ldots, C_m$ end up at clustered at position $1 - \frac{\rho}{n-\nu}$.
4) We apply the construction recursively to the n vertices, which uses up a quantity of s-energy equal to $\big(1 - \frac{\rho}{\nu} - \frac{\rho}{n-\nu}\big)^s F(n,m)$.

Putting all the energetic contributions together, we find that, for constants $b', c > 0$,

$$\begin{aligned}F(n,m) &\ge 1 + \frac{2b\nu^{2-s}}{s\rho^{1-s}} + \Big(\frac{\rho}{\nu}\Big)^s F(n-\nu, m-1) \\ &\quad + \Big(1 - \frac{2\rho}{\nu}\Big)^s F(n,m) \\ &\ge \frac{b'}{s\rho^{1-s}} \Big(\frac{n}{m}\Big)^{1-s} F(n-\nu, m-1) \\ &\ge \Big(\frac{c}{s\rho^{1-s}}\Big)^m \Big(\frac{n}{m}\Big)^{(1-s)m+1}.\end{aligned}$$

THEOREM 2.4. There exist reversible agreement systems with initial diameter equal to 1 whose s-energy is at least $\big(c/s\rho^{1-s}\big)^m (n/m)^{(1-s)m+1}$, for constant $c > 0$. The number of vertices is n and the number of connected components is bounded by m; furthermore, all positive entries in the stochastic matrices are at least ρ.

Our lower bound constructions assume a unit diameter at time 0. Since $\mathcal{E}_{m,s}$ is defined for unit variance systems, we must scale the bound appropriately to compare the lower bound with Theorem 2.1. We have $q = \mathbf{1}/\rho$, so the variance is at most n/ρ and we scale the lower bound by $(\rho/n)^{s/2}$.

III. THE CONVERGENCE RATE OF FLOCKING

We rewrite the map of the velocity dynamics (1) in matrix form, $v(t+1) = P_t v(t)$, for $t \ge 0$, where $v(t)$ is an n-by-3 matrix with each row indicating a velocity vector. We have $P_t = Q^{-1} M_t$, where: $Q = \mathrm{diag}(q)$; $q_i = 1/a_i$; $(M_t)_{ij} = 1/a_i - |N_i(t)|$ if $i = j$ and 1 else. Note that $\bar{q} := \|q\|_1^{-1} q$ is the joint stationary distribution and $q = M_t \mathbf{1} \le \mathbf{1}/\rho$, where $\rho := \min_i a_i \in (0, 1/2]$. This shows that each one of the three coordinates provides its own reversible agreement system $\mathcal{S}_j$ $(j = 1, 2, 3)$. The only difference between the systems is their initial states. Recall that the s-energy of any such system is defined as $\sum_t E_{s,t}$, where $E_{s,t} = \sum_i l_i(t)^s$ and $l_i(t)$ is the length of the i-th block at time t. Let m be the maximum number of flocks and $N_{m,\alpha}$ the number of times t at which some block length $l_i(t)$ from at least one of $\mathcal{S}_j$ $(j = 1, 2, 3)$ exceeds α. For $0 < \alpha < 1$, we have $N_{m,\alpha} \le \inf_{s \in (0,1]} 3\alpha^{-s} \mathcal{E}_{m,s}$. Our assumption that $\|v_i(0)\|^2 \le \rho/n$ for all birds i implies that the three systems have variance at most one. By Theorem 2.1, for some (other) constant $c > 0$, setting $s = 1/\log(1/\alpha)$ yields

$$N_{m,\alpha} \le \Big(\frac{cn^2}{\rho} \log \frac{1}{\alpha}\Big)^m. \qquad (4)$$

[3]This can be relaxed with a simple padding argument we may omit.

[4]To keep the time finite, we can always force completion in a single step once the agents are sufficiently close to each other and use a limiting argument.

A. Single-Flock Dynamics

Between two consecutive switches (ie, edge changes), the flocking networks consists of fixed non-interacting flocks. We can analyze them separately. Without loss of generality, assume that G_t is a connected, time-invariant graph. We focus on system $\mathcal{S}_1$ for convenience. It consists of a single block at each timestep, so the s-energy is of the form $\sum_t \Delta_t^s$, where Δ_t is the diameter of the system at time $t \geq 0$. The diameter can never grow, so by the same argument leading to (4), we know that $\Delta_t \leq \alpha$ for any $t \geq \inf_{s\in(0,1]} \alpha^{-s}\mathcal{E}_{1,s}$. It follows that $\Delta_t \leq e^{-a\rho t/n^2}$, for constant $a > 0$. Recall that $x(t)$ and $v(t)$ are n-by-3 matrices; denote their first column by $y(t)$ and $w(t)$, respectively. Write $y(0) = y$, $w(0) = w$, and $P_t = P$. The vector $w(t) = P^t w$ tends to $(\bar{q}^T w)\mathbf{1}$. Since its coordinates lie in an interval of width Δ_t, it follows that $w(t) = (\bar{q}^T w)\mathbf{1} + \zeta(t)$, where $\|\zeta(t)\|_\infty \leq \Delta_t \leq e^{-a\rho t/n^2}$. Thus, for some $\gamma, \eta_t \in \mathbb{R}^n$,

$$y(t) = y + \sum_{k=1}^{t} w(k) = y + t(\bar{q}^T w)\mathbf{1} + \sum_{k=1}^{t} \zeta(k) = \beta t + \gamma + \eta_t,$$

where $\beta = (\bar{q}^T w)\mathbf{1}$, $\gamma = y + \sum_{k=1}^{\infty} \zeta(k)$, and $\|\eta_t\|_\infty \leq e^{-b\rho t/n^2}$, for constant $b > 0$. The same holds true for the other two coordinates, so the birds in the flock fly parallel to a straight line with a deviation from their asymptotic line vanishing exponentially fast. If so desired, it is straightforward to lock the flocks by stipulating that no two birds can lose an edge between them unless their velocities exceed a small threshold θ; because of the exponential convergence rate, choosing θ small enough ensures that two birds i and j adjacent in a flock may exceed distance r by only a tiny amount.

B. Flock Fusion

To bound the relaxation time, we begin with an intriguing geometric fact: Far enough into the future, two birds can only come close to each other if their velocities are nearly identical. In other words, encounters at large angles of attack cannot occur over a long time horizon. We begin with a technical lemma: A stationary observer positioned at the initial location of a bird sees that bird move less and less over time; this is because the bird flies increasingly in the direction of the line of sight.

LEMMA 3.1. For constant c and any $t > 1$, $\left\|v_i(t) - \frac{1}{t}\big(x_i(t) - x_i(0)\big)\right\| \leq \big(cn^2/\rho\big)^{m+2}(\log t)/t$, for any i.

Proof. For notational convenience, we set $i = 1$ and we denote by $y_j(t)$ (resp. $w_j(t)$) the first coordinate of $x_j(t)$ (resp. $v_j(t)$). The line-of-sight direction of bird 1 is given by $\frac{1}{t}\big(x_1(t) - x_1(0)\big)$. Along the first coordinate axis, this gives

$$u := \frac{1}{t}\big(y_1(t) - y_1(0)\big) = \frac{1}{t}\sum_{k=1}^{t} w_1(k). \tag{5}$$

Consider the difference $\delta := u - w_1(t)$. We can define the corresponding quantity for each of the other two directions and assume that δ has the largest absolute value among the three of them. By symmetry, we can also assume that $\delta \geq 0$; therefore

$$\left\|v_1(t) - \tfrac{1}{t}\big(x_1(t) - x_1(0)\big)\right\| \leq \sqrt{3}\,\delta. \tag{6}$$

The proof of the lemma rests on showing that, if δ is too large, some bird l must be at a distance greater than 1 from bird 1 at time 0, which has been ruled out. To identify the far-away bird l, we start with $l = 1$ at time t, and we trace the evolution of its flock backwards in time, always trying to move away from bird 1, if necessary by switching bird l with a neighbor. This is possible because of two properties, at least one of which holds at any time k: (i) bird l flies nearly straight in the time interval $[k, k+1]$; or (ii) bird l is adjacent to a bird l' whose velocity points in a favorable direction. In the latter case, we switch focus from l to l'.

The s-energy plays the key role in putting numbers behind these properties. For this reason, we define $\mu_l(k)$ as the length of the block of $\mathcal{S}_1$ containing $w_l(k)$ with respect to the flocking network G_k. Note that $\mu_l(k)$ is the length of an interval that contains the numbers $w_j(k)$ for all the birds j in the flock of bird l at time k. We define the sequence of velocities $\bar{w}(k) = w_l(k)$, for $k = t, t-1, \ldots, 1$ and $l = l(k)$. Fix some small α $(0 < \alpha \leq \varepsilon_o)$.

[1] $\bar{w}(t) \leftarrow w_1(t)$ and $l \leftarrow 1$
[2] **for** $k = t-1, \ldots, 1$
[3] **if** $\mu_l(k) > \alpha$ **then**
$l \leftarrow \operatorname{argmin}\{\, w_j(k) \mid j \in N_l(k) \,\}$
[4] $\bar{w}(k) \leftarrow w_l(k)$

Perhaps the best way to understand the algorithm is first to imagine that the conditional in step [3] never holds: In that case, $l = 1$ throughout and we are simply tracing the backward evolution of bird 1. Step [3] aims to catch the instances where the reverse trajectory inches excessively toward the initial position of bird 1. When that happens, $|w_l(k+1) - w_l(k)|$ is large, hence so is $\mu_l(k)$, and step [3] kicks in. We exploit the fact that $w_l(k+1)$ is a convex combination of $\{w_j(k) \mid j \in N_l(k)\}$ to update the current bird l to a "better" one. Using summation by parts, we find that

$$\sum_{k=1}^{t} \bar{w}(k) = t\bar{w}(t) - \sum_{k=1}^{t-1} k\big(\bar{w}(k+1) - \bar{w}(k)\big). \tag{7}$$

Let R be the set of times k that pass the test in step [3] and S the set of switches (ie, network changes). An edge creation entails a block of length $\varepsilon_o/\sqrt{3}$ or more in at least one of $\mathcal{S}_j$ $(j = 1, 2, 3)$. The steps witnessing edge deletions outnumber those seeing edge creations by at most a factor of $\binom{n}{2}$. Let I be the time interval between two consecutive switches. Each flock remains invariant during I; thus $|R \cap I| \leq N_{1,\alpha}$; hence

$$|S| \leq n^2 N_{m,\varepsilon_o/\sqrt{3}} \qquad \text{and} \qquad |R| \leq (N_{1,\alpha} + 1)|S|. \tag{8}$$

Because of the single-flock invariance, the diameter of $\mathcal{S}_1$ during I can never increase; therefore $J = I \setminus R$ consists of a single time interval. If $k \in J$, then $|\bar{w}(k+1)-\bar{w}(k)| = |w_l(k+1) - w_l(k)| \le \mu_l(k) \le \alpha$ and, by Theorem 2.1, $\sum_{k\in J} |\bar{w}(k+1) - \bar{w}(k)| \le \sum_{k\in J} E_{1,k} \le \alpha\, \mathcal{E}_{1,1} = O(\alpha n^2/\rho)$; hence

$$\sum_{k\in\{1,\dots,t-1\}\setminus R} \left|\bar{w}(k+1) - \bar{w}(k)\right| = O(\alpha n^2|S|/\rho). \qquad (9)$$

Let l' be the value of l in the final assignment $\bar{w}_l(1) \leftarrow w_l(1)$ in step [4]. Since $\bar{w}(k+1) \ge \bar{w}(k)$ for $k \in R$ and $\bar{w}(t) = w_1(t)$, it follows from (7, 9) and $r \le 1$ that

$$\begin{aligned} y_{l'}(0) - y_1(0) &\ge \big(y_1(t) - y_1(0)\big) + \big(y_{l'}(0) - y_1(t)\big) \\ &\ge tu - \sum_{k=1}^{t} \bar{w}(k) - r|R| \\ &\ge t\delta + \sum_{k=1}^{t-1} k\big(\bar{w}(k+1) - \bar{w}(k)\big) - |R| \\ &\ge t\delta - O(t\alpha n^2|S|/\rho) - |R|. \end{aligned} \qquad (10)$$

We set $\alpha = \varepsilon_o/t$. Noting that $y_{l'}(0) - y_1(0) \le 1$, the lemma follows from (4, 6, 8) and

$$\delta \le n^{2(m+2)} \left(\frac{b}{\rho}\right)^{m+1} \left(\log \frac{1}{\varepsilon_o}\right)^m \frac{\log(t/\varepsilon_o)}{t}.$$

for constant $b > 0$. □

C. Stabilization

By Lemma 3.1, for a large enough constant $c = c(\varepsilon_o)$, after time $t > t_o := \big(cn^2/\rho\big)^{m+2} \log(n/\rho)$, no bird's velocity differs from its line-of-sight vector $u_i = \frac{1}{t}\big(x_i(t) - x_i(0)\big)$ by a vector longer than $\varepsilon_o/3$. Suppose that birds i and j are within distance r of each other. By the triangular inequality, $\|u_i - u_j\| \le \frac{1}{t}\|x_i(t) - x_j(t)\| + \frac{1}{t}\|x_i(0) - x_j(0)\| \le (1+r)/t$; therefore,

$$\|v_i(t) - v_j(t)\| \le \|v_i(t) - u_i\| + \|u_i - u_j\| + \|v_j(t) - u_j\| \le \varepsilon_o.$$

This implies that each flock is time-invariant past time t_o. The birds within each flock align their velocities exponentially fast. In view of §III-A, this proves Theorem 1.1. □

IV. DISTRIBUTED MOTION COORDINATION

In [13] Sugihara and Suzuki introduced an interesting model of pattern formation in a swarm of robots. In their model, the robots can communicate anonymously and adjust their positions accordingly. Assume that their goal is to align themselves along a line segment ab. Two robots position themselves manually at the endpoints of the segment while the others attempt to reach ab by linking with their right/left neighbors and averaging their positions iteratively. This setup creates a polygonal line $u_1 = a, u_2, \dots, u_{n-1}, u_n = b$, where u_i is the position (x_i, y_i) of robot i. The polygonal line converges to ab in the limit. We use the s-energy bounds to evaluate the convergence time of the robots. We actually prove a stronger result by generalizing the model in two ways: (i) we consider the case of an arbitrary communication network of robots in 3D, with a subset of vertices pinned to a fixed plane; (ii) the network suffers from stochastic edge failures. Our model trivially reduces to Sugihara and Suzuki's by projection. Allowing stochastic failures to their motion coordination model is novel.

Let G be a connected (undirected) graph with n vertices labeled in $[n]$; and let the *communication weights* $a_1, \dots, a_n$ be n positive reals such that $a_i < 1/(d_i+1)$, where d_i is the degree of vertex i. We define $d = \max d_i$ and $\rho = \min a_i$. For any $t \ge 0$, we define G_t by deleting each edge of G with probability $1-p$. We define a (random) stochastic matrix P_t for G_t as follows:

1) Initialize $P_t = 0$;
2) If (i,j) is an edge of G_t, we set $(P_t)_{ij} = a_i$ and $(P_t)_{ji} = a_j$.
3) $(P_t)_{ii} = 1 - \sum_{j(j\neq i)} (P_t)_{ij}$, for all i.

Note that every positive entry of P_t is at least ρ. We embed G in $\mathbb{R}^3$ and pin a subset R of r vertices to a fixed plane. We fix the scale by assuming that the embedding lies in the unit cube $[0,1]^3$. Without loss of generality, we choose the plane $X = 0$. To ensure the immobility of the r vertices, we can set $a_i = 0$ for each $i \in R$. Equivalently, we use *symmetrization* [3] by attaching to R a copy of G and initializing the embedding of the two copies as mirror-image reflections about $X = 0$; note that the resulting graph has $\nu = 2n - r$ vertices. The sequence $(P_t)_{t\ge 0}$ is defined by picking a random G_t (as defined above) at each step iid.

The vertices of R are embedded in the plane $X = 0$ at time 0, where, by symmetry, they reside permanently. To prove the convergence of the ν points to the plane, it suffices to focus on the dynamics along the X-axis. Given $x(0) \in [-1,1]^\nu$, we have $x(t+1) = P_t x(t)$. This gives us a reversible agreement system. Using the notation from §II and Lemma 2.2, we have $q = (1/a_1, \dots, 1/a_\nu)$. Since G is connected, there is a path π connecting the leftmost to the rightmost vertex along the X-axis. By the Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbb{E}\, D_t &\ge \mathbb{E} \sum_{i=1}^{\nu} \max_{j:(i,j)\in G_t} \delta_{ij}^2 \ge \sum_{i=1}^{\nu} \sum_{j:(i,j)\in G} p\delta_{ij}^2/d_i \\ &\ge \frac{p}{d\nu}\Big(\sum_{(i,j)\in\pi} |\delta_{ij}|\Big)^2 \ge \frac{\rho p}{d\nu^2}\|x\|_q^2 \end{aligned}$$

where $\delta_{ij} = \delta_{ij}(t) = x_i(t) - x_j(t)$. It follows from Lemma 2.2 that, for $c := \rho p/(2d\nu^2)$,

$$\mathbb{E}\,\|Px\|_q^2 \le \|x\|_q^2 - \frac{1}{2}\,\mathbb{E}\, D_t \le (1-c)\|x\|_q^2.$$

By Markov's inequality,

$$\Pr\left[\|Px\|_q^2 \ge \Big(1 - \frac{c}{3}\Big)\|x\|_q^2\right] \le \frac{\mathbb{E}\,\|Px\|_q^2}{(1-c/3)\|x\|_q^2} \le 1 - \frac{c}{2}.$$

Let $l_1, \dots, l_k$ be the lengths of the blocks formed by the edges of G_t embedded along the X-axis.[5] In a slight variant, we define the s-energy $E_s = \sum_{t\ge 0} E_{s,t}$, where $E_{s,t} = \max_{i=1}^{k} l_i^s$. We denote by W_s the maximum expected s-energy, where the maximum is taken over all initial positions with variance

[5]Recall that the blocks are the intervals formed by the union of the embedded edges of G_t.

$\|x\|_q^2 \leq \nu/\rho$ (see §II-A for definitions). Since the vertices are embedded in $[-1,1]$ with symmetry about the origin, this applies to the case at hand. By Cauchy-Schwarz, we see that the diameter is at most $\sqrt{2}\,\|x\|_q \leq \sqrt{2\nu/\rho}$. By scaling invariance, we have the following recurrence relation:

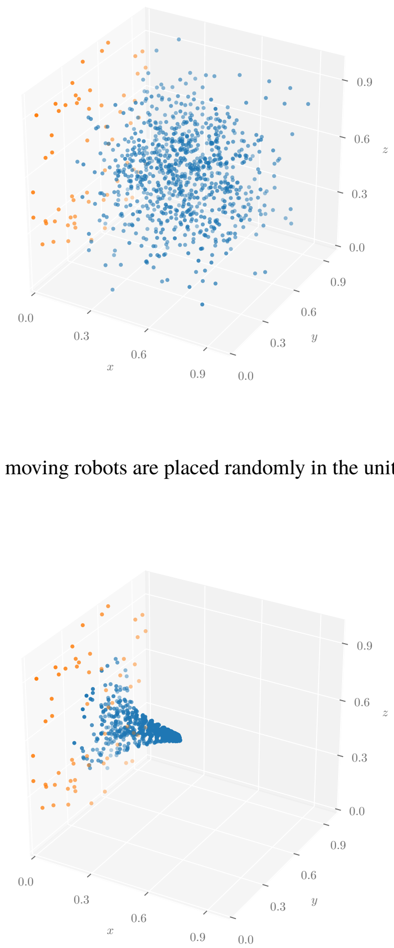
$$\begin{aligned} W_s &\leq \left(\frac{2\nu}{\rho}\right)^{s/2} + \frac{c}{2}\left(1-\frac{c}{3}\right)^{s/2} W_s + \left(1-\frac{c}{2}\right) W_s \\ &\leq \frac{2^{s+1}\nu^{s/2}}{c\rho^{s/2}\left(1-(1-c/3)^{s/2}\right)} \\ &= O\left(\frac{\nu^{s/2}}{sc^2\rho^{s/2}}\right) = O\left(\frac{dn^2}{\rho p}\right)^2 \frac{(n/\rho)^{s/2}}{s}. \end{aligned}$$

Let N_α be the number of times t at which some block length $l_i(t)$ exceeds α. For $0 < \alpha < 1$, we have $\mathbb{E}\,N_\alpha \leq \inf_{s\in(0,1]} \alpha^{-s} W_s$. Setting $s = 1/\log(n/\rho\alpha^2)$ yields

$$\mathbb{E}\,N_\alpha = O\left(\frac{dn^2}{\rho p}\right)^2 \log\frac{n}{\rho\alpha}.$$

Let K_α be the number of times t at which there exists an edge $(i,j) \in G_t$ whose length $|\delta_{ij}(t)|$ exceeds α; obviously, $K_\alpha \leq N_\alpha$. Let T_α be the last time at which the diameter of the system exceeds α. For each $t \leq T_\alpha$, being a connected graph, G must include an edge (i,j) whose length $|\delta_{ij}(t)|$ exceeds α/ν. That edge belongs to G_t with probability p; therefore $\mathbb{E}\,K_{\alpha/\nu} \geq p\,\mathbb{E}\,T_\alpha$; hence $\mathbb{E}\,T_\alpha \leq \frac{1}{p}\mathbb{E}\,N_{\alpha/\nu}$.

Theorem 4.1. The robots align themselves within distance $\epsilon < 1$ of a fixed plane in expected time $O\big(d^2n^4/p^3\rho^2\big)\log(n/\rho\epsilon)$, where d is the maximum degree of the underlying communication network, n is the number of robots, $1-p$ is the probability of edge failure, and ρ is the smallest communication weight.



(a) Initially, moving robots are placed randomly in the unit cube.

(b) Positions of moving robots after 200 steps.

Fig. 2. Simulation of a network of 900 robots in 3D with 60 of them (orange dots) pinned to a fixed plane $X = 0$ and the rest (blue dots) are moving. The underlying network G is a 30-by-30 grid graph, with edge failure probability equal to 0.3. The 60 nodes on two opposite sides of the grid are pinned to $X = 0$.

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